Rate of Change and Slope

Section 5-1
Goals

Goal

• To find rates of change from tables.
• To find slope.
Vocabulary

• Rate of Change
• Slope
Definition

• *Rate of Change* – a ratio that compares the amount of change in a dependent variable to the amount of change in an independent variable.

\[
\text{rate of change} = \frac{\text{change in dependent variable (y)}}{\text{change in independent variable (x)}}
\]

• The rates of change for a set of data may vary or they may be constant.
Example:

Determine whether the rates of change are constant or variable.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>16</td>
</tr>
</tbody>
</table>

Find the difference between consecutive data points.

Find each ratio of change in y to change in x.

\[
\frac{2}{1} = 2 \quad \frac{4}{2} = 2 \quad \frac{4}{2} = 2 \quad \frac{6}{3} = 2
\]

The rates of change are constant.
Determine whether the rates of change are constant or variable.

B. Find the difference between consecutive data points.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>3</td>
</tr>
</tbody>
</table>

Find each ratio of change in y to change in x.

\[
\frac{2}{2} = 1 \quad \frac{4}{1} = 4 \quad \frac{0}{2} = 0 \quad \frac{3}{3} = 1
\]

The rates of change are variable.
Your Turn:

Determine whether the rates of change are constant or variable.

A.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>11</td>
</tr>
</tbody>
</table>

Find the difference between consecutive data points.

Find each ratio of change in y to change in x.

\[
\frac{2}{1} = 2 \\
\frac{2}{1} = 2 \\
\frac{2}{1} = 2 \\
\frac{4}{1} = 4
\]

The rates of change are variable.
Your Turn:
Determine whether the rates of change are constant or variable.

B. Find the difference between consecutive data points.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>4</th>
<th>6</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>12</td>
<td>18</td>
</tr>
</tbody>
</table>

Find the difference between consecutive data points.

Find each ratio of change in y to change in x.

\[
\frac{2}{1} = 2 \quad \frac{6}{3} = 2 \quad \frac{4}{2} = 2 \quad \frac{6}{3} = 2
\]

The rates of change are constant.
Example: Application

The table shows the average temperature (°F) for five months in a certain city. Find the rate of change for each time period. During which time period did the temperature increase at the fastest rate?

<table>
<thead>
<tr>
<th>Month</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp. (°F)</td>
<td>56</td>
<td>56</td>
<td>63</td>
<td>71</td>
<td>72</td>
</tr>
</tbody>
</table>

Step 1 Identify the dependent and independent variables.

dependent: temperature
independent: month
Step 2 Find the rates of change.

<table>
<thead>
<tr>
<th>Month</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp. (°F)</td>
<td>56</td>
<td>56</td>
<td>63</td>
<td>71</td>
<td>72</td>
</tr>
</tbody>
</table>

\[
\text{change in temp.} = \frac{56 - 56}{3 - 2} = \frac{0}{1} = 0 \quad \text{0° mo}
\]

\[
\frac{63 - 56}{5 - 3} = \frac{7}{2} = 3.5 \quad \text{3.5° mo}
\]

\[
\frac{71 - 63}{7 - 5} = \frac{8}{2} = 4 \quad \text{4° mo}
\]

\[
\frac{72 - 71}{8 - 7} = \frac{1}{1} = 1 \quad \text{1° mo}
\]

The temperature increased at the greatest rate from month 5 to month 7.
Your Turn:

The table shows the balance of a bank account on different days of the month. Find the rate of change during each time interval. During which time interval did the balance decrease at the greatest rate?

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>6</th>
<th>16</th>
<th>22</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance ($)</td>
<td>550</td>
<td>285</td>
<td>210</td>
<td>210</td>
<td>175</td>
</tr>
</tbody>
</table>

Step 1 Identify the dependent and independent variables.

dependent: balance
independent: day
Step 2 Find the rates of change.

1 to 6
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{285 - 550}{6 - 1} = \frac{-265}{5} = -53 \frac{-53}{\text{day}}
\]

6 to 16
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{210 - 285}{16 - 6} = \frac{-75}{10} = -7.5 \frac{-7.5}{\text{day}}
\]

16 to 22
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{210 - 210}{22 - 16} = \frac{0}{6} = 0 \frac{0}{\text{day}}
\]

22 to 30
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{175 - 210}{30 - 22} = \frac{-35}{8} = -4.375 \frac{-4.375}{\text{day}}
\]

The balance declined at the greatest rate from day 1 to day 6.
Rates of Change on a Graph

To show *rates of change* on a graph, plot the data points and connect them with line segments.

Remember!

Data in a graph is always read from left to right.
Example: Rates of Change from a Graph

Graph the data from the first Example and show the rates of change.

<table>
<thead>
<tr>
<th>Month</th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp. (°F)</td>
<td>56</td>
<td>56</td>
<td>63</td>
<td>71</td>
<td>72</td>
</tr>
</tbody>
</table>

Graph the ordered pairs. The vertical segments show the changes in the dependent variable, and the horizontal segments show the changes in the independent variable.

Notice that the greatest rate of change is represented by the steepest of the red line segments.
Example: Continued

\[
\begin{align*}
\text{2 to 3} & \quad \frac{\text{change in temp.}}{\text{change in mo.}} = \frac{56 - 56}{3 - 2} = \frac{0}{1} = 0 \quad 0^\circ \text{mo} \\
\text{3 to 5} & \quad \frac{\text{change in temp.}}{\text{change in mo.}} = \frac{63 - 56}{5 - 3} = \frac{7}{2} = 3.5 \quad 3.5^\circ \text{mo} \\
\text{5 to 7} & \quad \frac{\text{change in temp.}}{\text{change in mo.}} = \frac{71 - 63}{7 - 5} = \frac{8}{2} = 4 \quad 4^\circ \text{mo} \\
\text{7 to 8} & \quad \frac{\text{change in temp.}}{\text{change in mo.}} = \frac{72 - 71}{8 - 7} = \frac{1}{1} = 1 \quad 1^\circ \text{mo}
\end{align*}
\]

Also notice that between months 2 to 3, when the balance did not change, the line segment is horizontal.
Your Turn:

Graph the data from the last your turn and show the rates of change.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>6</th>
<th>16</th>
<th>22</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance ($)</td>
<td>550</td>
<td>285</td>
<td>210</td>
<td>210</td>
<td>175</td>
</tr>
</tbody>
</table>

Graph the ordered pairs. The vertical segments show the changes in the dependent variable, and the horizontal segments show the changes in the independent variable.

Notice that the greatest rate of change is represented by the steepest of the red line segments.
Your Turn: Continued

1 to 6
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{285 - 550}{6 - 1} = \frac{-265}{5} = -53 \text{ per day}
\]

6 to 16
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{210 - 285}{16 - 6} = \frac{-75}{10} = -7.5 \text{ per day}
\]

16 to 22
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{210 - 6}{22 - 16} = \frac{0}{6} = 0 \text{ per day}
\]

22 to 30
\[
\frac{\text{change in balance}}{\text{change in days}} = \frac{175 - 210}{30 - 22} = \frac{-35}{8} = -4.375 \text{ per day}
\]

Also notice that between days 16 to 22, when the balance did not change, the line segment is horizontal.
• The constant rate of change of a line is called the **slope of the line**.

• **Slope** – Is a measure of the steepness of a line.
  – The steeper the line, the greater the slope.
  – Notation: $m$ stands for slope.
  – For any two points, the slope of the line connecting them can be determined by dividing how much the line rises vertically by how much it runs horizontally.
Slope

\[ \text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}} \]

In the diagram:

- The points (-5, 3) and (2, -1) are given.
- The slope \( m \) is calculated as \( m = \frac{4}{-7} = -\frac{4}{7} \).
- The rise is from (2, -1) to (0, 0), which is 4 units.
- The run is from (-5, 3) to (0, 0), which is 7 units.
Slope

**Slope of a Line**

The **rise** is the difference in the **y-values** of two points on a line.

The **run** is the difference in the **x-values** of two points on a line.

The **slope** of a line is the ratio of rise to run for any two points on the line.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$$

(Remember that **y** is the **dependent variable** and **x** is the **independent variable**.)
Example: Slope from a Graph

Find the slope of the line.

Begin at one point and count vertically to find the rise.

Then count horizontally to the second point to find the run.

It does not matter which point you start with. The slope is the same.

\[
\text{Slope} = \frac{-3}{9} = -\frac{1}{3}
\]

\[
\text{Slope} = \frac{3}{-9} = -\frac{1}{3}
\]
Your Turn:

Find the slope of the line that contains (0, –3) and (5, –5).

Begin at one point and count vertically to find rise.

Then count horizontally to the second point to find the run.

It does not matter which point you start with. The slope is the same.

\[ \text{Slope} = \frac{-2}{5} = -\frac{2}{5} \]
Your Turn:

Find the slope of the line.

Begin at one point and count vertically to find the rise.

Then count horizontally to the second point to find the run.

slope = \frac{2}{4} = \frac{1}{2}

The slope of the line is \frac{1}{2}.
Then count horizontally to the second point to find the run.

The slope of the line is $1$. 

Your Turn:

Find the slope of the line.
Find the slope of each line.

A.  
\[
\frac{\text{rise}}{\text{run}} = \frac{6}{0}
\]
You cannot divide by 0

The slope is undefined.

B.  
\[
\frac{\text{rise}}{\text{run}} = \frac{0}{12} = 0
\]

The slope is 0.
Your Turn:

Find the slope of each line.

A. 

\[
\frac{\text{rise}}{\text{run}} = \frac{-8}{0}
\]

You cannot divide by 0.

The slope is undefined.

B. 

\[
\frac{\text{rise}}{\text{run}} = \frac{0}{4} = 0
\]

The slope is 0.
As shown in the previous examples, slope can be positive, negative, zero or undefined. You can tell which of these is the case by looking at a graph of a line—you do not need to calculate the slope.

<table>
<thead>
<tr>
<th>Positive Slope</th>
<th>Negative Slope</th>
<th>Zero Slope</th>
<th>Undefined Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Positive Slope Diagram" /></td>
<td><img src="image" alt="Negative Slope Diagram" /></td>
<td><img src="image" alt="Zero Slope Diagram" /></td>
<td><img src="image" alt="Undefined Slope Diagram" /></td>
</tr>
<tr>
<td>Line rises from left to right.</td>
<td>Line falls from left to right.</td>
<td>Horizontal line</td>
<td>Vertical line</td>
</tr>
</tbody>
</table>
Example:

Tell whether the slope of each line is positive, negative, zero or undefined.

A. The line rises from left to right.
   The slope is positive.

B. The line falls from left to right.
   The slope is negative.
Tell whether the slope of each line is positive, negative, zero or undefined.

a. The line rises from left to right. The slope is positive. The line is vertical. The slope is undefined.

b. The line rises from left to right. The slope is positive.
## Comparing Slopes

<table>
<thead>
<tr>
<th>Comparing Slopes</th>
</tr>
</thead>
<tbody>
<tr>
<td>The line with slope 4 is steeper than the line with slope 1/2.</td>
</tr>
<tr>
<td>The line with slope −2 is steeper than the line with slope −1.</td>
</tr>
<tr>
<td>The line with slope −3 is steeper than the line with slope 3/4.</td>
</tr>
</tbody>
</table>

\[
|4| > \left| \frac{1}{2} \right| \\
|-2| > |-1| \\
|-3| > \left| \frac{3}{4} \right|
\]
Slope Formula

Recall that lines have constant slope. For a line on the coordinate plane, slope is the following ratio:

\[
\frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{change in } y}{\text{change in } x}
\]
Slope Formula

If you know any two points on a line, you can find the slope of the line without graphing. The slope of a line through the points \((x_1, y_1)\) and \((x_2, y_2)\) is as follows:

\[
\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}
\]

When finding slope using the ratio above, it does not matter which point you choose for \((x_1, y_1)\) and which point you choose for \((x_2, y_2)\).
Slope Formula

Key Concept  The Slope Formula

\[
\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 - x_1 \neq 0
\]

The x-coordinate you use first in the denominator must belong to the same ordered pair as the y-coordinate you use first in the numerator.
Example:

Find the slope of the line that passes through \((-2, -3)\) and \((4, 6)\).

Let \((x_1, y_1)\) be \((-2, -3)\) and \((x_2, y_2)\) be \((4, 6)\).

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-3)}{4 - (-2)}
\]

Substitute 6 for \(y_2\), \(-3\) for \(y_1\), 4 for \(x_2\), and \(-2\) for \(x_1\).

\[
= \frac{6 + 3}{4 + 2} = \frac{9}{6}
\]

Simplify.

\[
= \frac{3}{2}
\]

The slope of the line that passes through \((-2, -3)\) and \((4, 6)\) is \(\frac{3}{2}\).
Example:

Find the slope of the line that passes through \((1, 3)\) and \((2, 1)\).

Let \((x_1, y_1)\) be \((1, 3)\) and \((x_2, y_2)\) be \((2, 1)\).

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 3}{2 - 1}
\]

Substitute 1 for \(y_2\), 3 for \(y_1\), 2 for \(x_2\), and 1 for \(x_1\).

\[
= \frac{-2}{1} = -2
\]

Simplify.

The slope of the line that passes through \((1, 3)\) and \((2, 1)\) is \(-2\).
Example:

Find the slope of the line that passes through \((3, -2)\) and \((1, -2)\).

Let \((x_1, y_1)\) be \((3, -2)\) and \((x_2, y_2)\) be \((1, -2)\).

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - (-2)}{1 - 3}
\]

Substitute \(-2\) for \(y_2\), \(-2\) for \(y_1\), \(1\) for \(x_2\), and \(3\) for \(x_1\).

\[
= \frac{-2 + 2}{1 - 3}
\]

Rewrite subtraction as addition of the opposite.

\[
= \frac{0}{-2} = 0
\]

The slope of the line that passes through \((3, -2)\) and \((1, -2)\) is 0.
Find the slope of the line that passes through \((-4, -6)\) and \((2, 3)\).

Let \((x_1, y_1)\) be \((-4, -6)\) and \((x_2, y_2)\) be \((2, 3)\).

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-6)}{2 - (-4)}
\]

Substitute 3 for \(y_2\), \(-6\) for \(y_1\), 2 for \(x_2\), and \(-4\) for \(x_1\).

\[
= \frac{9}{6} = \frac{3}{2}
\]

The slope of the line that passes through \((-4, -6)\) and \((2, 3)\) is \(\frac{3}{2}\).
Your Turn:

Find the slope of the line that passes through (2, 4) and (3, 1).

Let \((x_1, y_1)\) be (2, 4) and \((x_2, y_2)\) be (3, 1).

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4}{3 - 2}
\]

\[
= \frac{-3}{1} = -3
\]

Substitute 1 for \(y_2\), 4 for \(y_1\), 3 for \(x_2\), and 2 for \(x_1\).

Simplify.

The slope of the line that passes through (2, 4) and (3, 1) is \(-3\).
Your Turn:

Find the slope of the line that passes through \((3, -2)\) and \((1, -4)\).

Let \((x_1, y_1)\) be \((3, -2)\) and \((x_2, y_2)\) be \((1, -4)\).

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - (-2)}{1 - 3}
\]

\[
= \frac{-4 + 2}{1 - 3}
\]

\[
= \frac{-2}{-2} = 1
\]

The slope of the line that passes through \((3, -2)\) and \((1, -4)\) is \(-2\).
Joke Time

• Why was the elephant standing on the marshmallow?
  • He didn’t want to fall in the hot chocolate!

• What would you say if someone took your playing cards?
  • dec-a-gon

• What kind of insect is good at math?
  • An account-ant