Real Numbers and the Number Line

Section 1-3
Goals

Goal

• To classify, graph, and compare real numbers.
• To find and estimate square roots.
Vocabulary

- Square Root
- Radicand
- Radical
- Perfect Square
- Set
- Element of a Set
- Subset
- Rational Numbers
- Natural Numbers
- Whole Numbers
- Integers
- Irrational Numbers
- Real Numbers
- Inequality
Square Roots

Opposite of squaring a number is taking the square root of a number.

A number $b$ is a square root of a number $a$ if $b^2 = a$.

In order to find a square root of $a$, you need a # that, when squared, equals $a$. 
$2^2 = 4$
The square root of 4 is 2
$3^2 = 9$
The square root of 9 is 3.
$4^2 = 16$
The square root of 16 is 4
$$5^2 = 25$$
The square root of 25 is 5
Principal Square Roots

Any positive number has two real square roots, one positive and one negative, $\sqrt{x}$ and $-\sqrt{x}$

$\sqrt{4} = 2$ and $-2$, since $2^2 = 4$ and $(-2)^2 = 4$

The principal (positive) square root is noted as

The negative square root is noted as
Radicand

*Radical expression* is an expression containing a radical sign.

*Radicand* is the expression under a radical sign.

Note that if the radicand of a square root is a negative number, the radical is NOT a real number.
Square roots of perfect square radicands simplify to rational numbers (numbers that can be written as a quotient of integers).

Square roots of numbers that are not perfect squares (like 7, 10, etc.) are irrational numbers.

IF REQUESTED, you can find a decimal approximation for these irrational numbers. Otherwise, leave them in radical form.
Perfect Squares

The terms of the following sequence:

1, 4, 9, 16, 25, 36, 49, 64, 81…

$1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2…$

These numbers are called the Perfect Squares.
The small number to the left of the root is the *index*. In a square root, the index is understood to be 2. In other words, $\sqrt{}$ is the same as $\sqrt[2]{}$. 
Roots

A number that is raised to the third power to form a product is a **cube root** of that product. The symbol $\sqrt[3]{}$ indicates a cube root. Since $2^3 = 8$, $\sqrt[3]{8} = 2$. Similarly, the symbol $\sqrt[4]{}$ indicates a fourth root: $2^4 = 16$, so $\sqrt[4]{16} = 2$. 
Example: Finding Roots

Find each root.

A. \( \sqrt{81} \)
\[
\sqrt{81} = \sqrt{9^2} = 9
\]

Think: What number squared equals 81?

B. \( -\sqrt{25} \)
\[
-\sqrt{25} = -\sqrt{5^2} = -5
\]

Think: What number squared equals 25?
Example: Finding Roots

Find the root.

C. \( \sqrt[3]{-216} \)

\[
\sqrt[3]{-216} = \sqrt[3]{(-6)^3}
\]

**Think:** What number cubed equals –216?

= –6

\((-6)(-6)(-6) = 36(-6) = -216\)
Your Turn:

Find each root.

a. \( \sqrt{4} \)
   \[ \sqrt{4} = \sqrt{2^2} \]
   \[ = 2 \]
   
   Think: What number squared equals 4?

b. \( -\sqrt{25} \)
   \[ -\sqrt{25} = -\sqrt{5^2} \]
   \[ = -5 \]

Think: What number squared equals 25?
Your Turn:

Find the root.

c. \(4\sqrt[4]{81}\)

\[
4\sqrt[4]{81} = 4\sqrt{3^4}
\]

\[
= 3
\]

Think: What number to the fourth power equals 81?
Example: Finding Roots of Fractions

Find the root.

A. \( \sqrt{\frac{49}{9}} \)

\[
\frac{7 \cdot 7}{3 \cdot 3} = \frac{49}{9}
\]

Think: What number squared equals \( \frac{49}{9} \)?

\[
\sqrt{\frac{49}{9}} = \frac{7}{3}
\]
Example: Finding Roots of Fractions

Find the root.

B. \[ \sqrt[3]{\frac{8}{125}} \]

\[ \frac{2 \cdot 2 \cdot 2}{5 \cdot 5 \cdot 5} = \frac{8}{125} \]

Think: What number cubed equals \( \frac{8}{125} \)?

\[ \sqrt[3]{\frac{8}{125}} = \frac{2}{5} \]
Example: Finding Roots of Fractions

Find the root.  

\[ \sqrt[3]{\frac{4}{25}} \]

C.  \[ -\sqrt[3]{\frac{4}{25}} \]

\[ -\left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = -\frac{4}{25} \]

Think: What number squared equals \[ \frac{4}{25} \]?

\[ -\sqrt{\frac{4}{25}} = -\frac{2}{5} \]
Your Turn:

Find the root.

a. $\sqrt{\frac{4}{9}}$

\[ \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9} \]

Think: What number squared equals \( \frac{4}{9} \)?

\[ \sqrt{\frac{4}{9}} = \frac{2}{3} \]
Your Turn:

Find the root.

b. \( \sqrt[3]{\frac{1}{8}} \)

\[
\frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}}{2} = \frac{1}{8}
\]

Think: What number cubed equals \( \frac{1}{8} \)?

\[
\sqrt[3]{\frac{1}{8}} = \frac{1}{2}
\]
Your Turn:

Find the root.

c. $\sqrt{\frac{4}{49}}$

Think: What number squared equals $\frac{4}{49}$?

$-\left(\frac{2}{7}\right)\left(\frac{2}{7}\right) = -\frac{4}{49}$

$-\sqrt{\frac{4}{49}} = -\frac{2}{7}$
Square roots of numbers that are not perfect squares, such as 15, are irrational numbers. A calculator can approximate the value of $\sqrt{15}$ as 3.872983346... Without a calculator, you can use square roots of perfect squares to help estimate the square roots of other numbers.
Example: Application

As part of her art project, Shonda will need to make a paper square covered in glitter. Her tube of glitter covers 13 in². Estimate to the nearest tenth the side length of a square with an area of 13 in².

Since the area of the square is 13 in², then each side of the square is $\sqrt{13}$ in. 13 is not a perfect square, so find two consecutive perfect squares that $\sqrt{13}$ is between: 9 and 16. $\sqrt{13}$ is between $\sqrt{9}$ and $\sqrt{16}$, or 3 and 4. Refine the estimate.
Example: Application Continued

Because $13$ is closer to $16$ than to $9$, $\sqrt{13}$ is closer to $4$ than to $3$.

You can use a guess-and-check method to estimate $\sqrt{13}$.
Example: Application Continued

Guess 3.6: $3.6^2 = 12.96$  
$\sqrt{13}$ is greater than 3.6. too low

Guess 3.7: $3.7^2 = 13.69$  
$\sqrt{13}$ is less than 3.7. too high

Because 13 is closer to 12.96 than to 13.69, $\sqrt{13}$ is closer to 3.6 than to 3.7.  $\sqrt{13} \approx 3.6$
The symbol $\approx$ means “is approximately equal to.”
What if...? Nancy decides to buy more wildflower seeds and now has enough to cover 26 ft$^2$. Estimate to the nearest tenth the side length of a square garden with an area of 26 ft$^2$.

Since the area of the square is 26 ft$^2$, then each side of the square is $\sqrt{26}$ ft. 26 is not a perfect square, so find two consecutive perfect squares that $\sqrt{26}$ is between: 25 and 36. $\sqrt{26}$ is between $\sqrt{25}$ and $\sqrt{36}$, or 5 and 6. Refine the estimate.
Solution Continued

5.0 \[5.0^2 = 25\] too low \[\sqrt{26} \text{ is greater than } 5.0.\]

5.1 \[5.1^2 = 26.01\] too high \[\sqrt{26} \text{ is less than } 5.1.\]

Since 5.0 is too low and 5.1 is too high, \(\sqrt{26}\) is between 5.0 and 5.1. Rounded to the nearest tenth, \(\sqrt{26} \approx 5.1.\)

The side length of the square garden is \(\sqrt{26} \approx 5.1\) ft.
Sets:

• A **set** is a collection of objects.
  – These objects can be anything: Letters, Shapes, People, Numbers, Desks, cars, etc.
  – Notation: Braces ‘{ }’, denote “The set of …”

• The objects in a set are called **elements of the set**.

• For example, if you define the set as all the fruit found in my refrigerator, then apple and orange would be elements or members of that set.

• A **subset** of a set consists of elements from the given set. A subset is part of another set.
Definitions: Number Sets

- **Natural numbers** are the counting numbers: 1, 2, 3, ...
- **Whole numbers** are the natural numbers and zero: 0, 1, 2, 3, ...
- **Integers** are whole numbers and their opposites: −3, −2, −1, 0, 1, 2, 3, ...
- **Rational numbers** can be expressed in the form \( \frac{a}{b} \), where \( a \) and \( b \) are both integers and \( b \neq 0 \): \( \frac{1}{2} \), \( \frac{7}{1} \), \( \frac{9}{10} \).
Definitions: Number Sets

- **Terminating decimals** are rational numbers in decimal form that have a finite number of digits: 1.5, 2.75, 4.0

- **Repeating decimals** are rational numbers in decimal form that have a block of one or more digits that repeat continuously: 1.3, 0.6, 2.14

- **Irrational numbers** cannot be expressed in the form a/b. They include square roots of whole numbers that are not perfect squares and nonterminating decimals that do not repeat: \( \sqrt{2}, \sqrt{11} \)
Rational or Not

1. $3.454545\ldots$  
   **Rational**

2. $1.23616161\ldots$  
   **Rational**

3. $0.1010010001\ldots$  
   **Irrational**

4. $0.34251$  
   **Rational**

5. $\pi$  
   **Irrational**
All numbers that can be represented on a number line are called **real numbers** and can be classified according to their characteristics.

![Number Sets Diagram]

- **Real Numbers**
  - **Rational Numbers** ($\mathbb{Q}$)
    - $\frac{27}{4}$
    - $-3$
    - $\frac{10}{11}$
  - **Integers** ($\mathbb{Z}$)
    - $-3$
    - $-2$
    - $0$
  - **Whole Numbers** ($\mathbb{W}$)
    - $-1$
    - $1$
    - $2$
    - $3$
    - $4.5$
    - $\frac{5}{9}$
  - **Natural Numbers** ($\mathbb{N}$)
    - $1$
    - $2$
    - $3$
  - **Irrational Numbers**
    - $\sqrt{17}$
    - $\sqrt{2}$
    - $\pi$
    - $e$
    - $-\sqrt{11}$
Number Sets - Notation

- **N** Natural Numbers - Set of positive integers
  \{1,2,3,…\}

- **W** Whole Numbers - Set of positive integers & zero
  \{0,1,2,3,…\}

- **Z** Set of integers
  \{0,±1,±2,±3,…\}

- **Q** Set of rational numbers
  \{x: x=a/b, b≠0 \cap a\in\mathbb{Z}, b\in\mathbb{Z}\}

- **\overline{Q}** Set of irrational numbers
  \{x: x is not rational\}

- **R** Set of real numbers
  \(-∞,∞\)
Example: State all numbers sets to which each number belongs?

1. $\frac{2}{3}$  
   1. Rational, real

2. $\sqrt{4}$  
   2. Natural, integer, rational, real

3. $\pi$  
   3. Irrational, real

4. -3  
   4. Integer, rational, real

5. $\sqrt{21}$  
   5. Irrational, real

6. 1.2525…  
   6. Rational, real
A *number line* is a line with marks on it that are placed at equal distances apart.

One mark on the number line is usually labeled zero and then each successive mark to the left or to the right of the zero represents a particular unit such as 1 or $\frac{1}{2}$.

On the number line above, each small mark represents $\frac{1}{2}$ unit and the larger marks represent 1 unit.
Rational Numbers on a Number Line

Integers

Whole Numbers

Negative numbers

Positive numbers

Zero is neither negative nor positive
Definition

• **Inequality** – a mathematical sentence that compares the values of two expressions using an inequality symbol..

• The symbols are:
  – $<$, less than
  – $\leq$, less than or equal to
  – $>$, Greater than
  – $\geq$, Greater than or equal to
Comparing the position of two numbers on the number line is done using inequalities.

\[ a < b \text{ means } a \text{ is } \text{to the left of} \ b \]
\[ a = b \text{ means } a \text{ and } b \text{ are at the same location} \]
\[ a > b \text{ means } a \text{ is } \text{to the right of} \ b \]

Inequalities can also be used to describe the sign of a real number.

\[ a > 0 \text{ is equivalent to } a \text{ is positive.} \]
\[ a < 0 \text{ is equivalent to } a \text{ is negative.} \]
Comparing Real Numbers

• We compare numbers in order by their location on the number line.

• Graph $-4$ and $-5$ on the number line. Then write two inequalities that compare the two numbers.

Since $-5$ is farther left, we say $-4 > -5$ or $-5 < -4$

• Put $-1, 4, -2, 1.5$ in increasing order

Left to right $-2, -1, 1.5, 4$
Your Turn:

- Write the following set of numbers in increasing order:

\[ -2.3, -4.8, 6.1, 3.5, -2.15, 0.25, 6.02 \]

\[ -4.8, -2.3, -2.15, 0.25, 3.5, 6.02, 6.1 \]
Comparing Real Numbers

• To compare real numbers rewrite all the numbers in decimal form.

• To convert a fraction to a decimal, **Divide the numerator by the denominator**

• Write each set of numbers in increasing order.

a. $\frac{2}{3}$, 0.3, $-\frac{5}{2}$, 0, 0.25, $\frac{2}{9}$

b. $5\frac{1}{2}$, $-3.8$, $\frac{9}{2}$, 4.8, $-\frac{6}{5}$

-3.8, $-\frac{6}{5}$, $\frac{9}{2}$, 4.8, $5\frac{1}{2}$

• YOU TRY c and d!

c. $-3$, -3.2, -3.15, -3.001, 3

d. $\frac{1}{3}$, $\frac{2}{7}$, $-\frac{5}{3}$, $-\frac{8}{3}$, $\frac{7}{9}$

-3.2, -3.15, -3.001, -3, 3
Example: Comparing Real Numbers

You can write a set of real numbers in order from greatest to least or from least to greatest.

To do so, find a decimal approximation for each number in the set and compare.

Write \(\frac{12}{5}, \sqrt{6}, 2.4, \text{ and } \frac{61}{25}\) in order from least to greatest. Write each number as a decimal.
Solution:

\[
\frac{12}{5}, \sqrt{6}, 2.4, \text{ and } \frac{61}{25}
\]

\[
\frac{12}{5} = 2.4
\]

\[
\sqrt{6} = 2.4494897\ldots \quad \text{or about } 2.4495
\]

\[
2.4 = 2.444444\ldots \quad \text{or about } 2.4444
\]

\[
\frac{61}{25} = 2.44
\]

2.4 < 2.44 < 2.4444 < 2.4495

Answer: The numbers arranged in order from least to greatest are \[
\frac{12}{5}, \frac{61}{25}, 2.4, \sqrt{6}.
\]
Your turn:

Write \( \frac{5}{2}, \sqrt{5}, 2.2, \text{ and } \frac{51}{20} \) in order from least to greatest.

Answer: \( 2.2, \sqrt{5}, \frac{5}{2}, \frac{255}{100} \)
Your Turn:

• What is the order of $\sqrt{9}$, 0.3, $-\frac{1}{4}$, $\sqrt{3}$, and $-4.25$ from least to greatest?

• Answer: $-4.25$, $-\frac{1}{4}$, 0.3, $\sqrt{3}$, $\sqrt{9}$
Joke Time

• Why do cows wear bells around their necks?
  • Because their horns don’t work.

• Where do you get virgin wool from?
  • Ugly sheep.

• Why did the fish blush?
  • Because it saw the ocean’s bottom.