Pre-Calculus
Solving Polynomial and Rational Inequalities – Two Methods

Example 1
Solve $x^2 - 11x + 28 \geq 0$.

If we can factor the quadratic expression on the left side of the inequality, then we apply the following arithmetic property:

$$AB \geq 0 \rightarrow (A \geq 0 \text{ and } B \geq 0) \text{ or } (A \leq 0 \text{ and } B \leq 0).$$

This says that the product of two factors will be positive if both of the factors are positive or both of the factors are negative. Thus:

$$x^2 - 11x + 28 \geq 0$$

$$(x - 4)(x - 7) \geq 0$$

$$(x - 4 \geq 0 \text{ and } x - 7 \geq 0) \quad \text{OR} \quad (x - 4 \leq 0 \text{ and } x - 7 \leq 0)$$

$$(x \geq 4 \text{ and } x \geq 7) \quad \text{OR} \quad (x \leq 4 \text{ and } x \leq 7)$$

$$x \geq 7 \quad \text{OR} \quad x \leq 4$$

The solution to the inequality $x^2 - 11x + 28 \geq 0$ is $x \leq 4$ or $x \geq 7$.

The next example has the same inequality being solved, but using a slightly different procedure – a sign analysis or sign chart procedure.
Example 2  Solve Using a Sign Analysis or Sign Chart Procedure

Solve \( x^2 - 11x + 28 \geq 0 \).

This procedure uses the same property as the previous example, but organizes the work in a little different manner and focuses only on the sign of the product of the factors.

Factor the polynomial expression completely: \( x^2 - 11x + 28 \geq 0 \) \( \rightarrow \) \( (x - 4)(x - 7) \geq 0 \)

Set each factor equal to zero and solve: \( x - 4 = 0 \rightarrow x = 4 \) \( x - 7 = 0 \rightarrow x = 7 \)

Locate the values found above to break the real number line into three intervals, as shown below. The values \( x = 4 \) and \( x = 7 \) will be included in the solution set to the inequality since the product must be greater than or equal to zero, so we will use solid dots on the number line at these locations.

We now choose one value of \( x \) in each of the intervals to test the sign of the product \( (x - 4)(x - 7) \). We will refer to this value as a test point.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Point</th>
<th>Sign of ( x - 4 )</th>
<th>Sign of ( x - 7 )</th>
<th>Sign of ( (x - 4)(x - 7) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 4 )</td>
<td>( x = 0 )</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( 4 &lt; x &lt; 7 )</td>
<td>( x = 5 )</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( x &gt; 7 )</td>
<td>( x = 10 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

Since the product must be non-negative to produce a solution to the inequality, we would take all intervals where the sign of the product was positive or equal to zero. Therefore, the solution to the inequality \( x^2 - 11x + 28 \geq 0 \) is \( x \leq 4 \) or \( x \geq 7 \).

You will likely not show this much work or detail in solving inequalities with this procedure. Remember that this example was completed thoroughly for demonstration purposes. In the following examples, the “work” will be substantially less, but the results will be the same – a quicker method for solving a polynomial inequality than using the first procedure.
Example 3
Solve \( x^3 + 3x^2 - 4x < 0 \) using the sign analysis procedure.

Factor completely:
\[
x^3 + 3x^2 - 4x < 0 \quad \rightarrow \quad x(x^2 + 3x - 4) < 0 \quad \rightarrow \quad x(x + 4)(x - 1) < 0
\]

Set factors equal to zero and solve:
\[
x = 0 \quad x + 4 = 0 \quad x - 1 = 0
\]
\[
x = 0 \quad x = -4 \quad x = 1
\]

Create subintervals on the real number line using the values found when setting the factors of the polynomial equal to zero and solving and perform the sign analysis using a test point and the factored form of the polynomial.

Test Point \( x = -6 \):
\((-)(-)(-)
\)

Test Point \( x = -2 \):
\((-)(+)(-)
\)

Test Point \( x = 0.5 \):
\((+)(+)(-)
\)

Test Point \( x = 3 \):
\((+)(+)(+)
\)

Since the inequality is “less than zero,” the product of the factors must be negative in order to produce a solution to the inequality. Thus, the solution set to the inequality \( x^3 + 3x^2 - 4x < 0 \), written in set-builder notation, is \( \{ x \mid x < -4 \text{ or } 0 < x < 1 \} \).
Example 4  Solving Rational Inequalities

Rational inequalities can also be solved using a sign analysis procedure. With rational inequalities, however, there is an additional area of consideration – values of $x$ that make the rational expression undefined. Consider the rational inequality below:

$$\frac{x^2 + 2x - 3}{x + 1} \geq 0$$

The rational expression is undefined whenever $x + 1 = 0$. Therefore, the value $x = -1$ cannot be included in our solution to this inequality. We will display this in our sign analysis procedure by placing an open circle at $x = -1$.

Now we to find the values of $x$ that will make the rational expression equal to 0. That is, we set the numerator equal to zero and solve. These values will be included in the solution set since the inequality specifies all values of $x$ that produce a value greater than or equal to 0. On the number line, we show these values as closed circles.

$$\frac{x^2 + 2x - 3}{x + 1} = 0 \quad \rightarrow \quad x^2 + 2x - 3 = 0 \quad \rightarrow \quad (x + 3)(x - 1) = 0$$

$$x = -3 \text{ or } x = 1$$

Complete the sign analysis procedure by choosing test points and determining the sign of the rational expression. Using the factored form of the rational expression will be easiest to determine the sign of the expression.

$$\frac{x^2 + 2x - 3}{x + 1} \geq 0 \quad \rightarrow \quad \frac{(x + 3)(x - 1)}{x + 1} \geq 0$$

<table>
<thead>
<tr>
<th>Test Point $x = -4$</th>
<th>Test Point $x = -2$</th>
<th>Test Point $x = 0$</th>
<th>Test Point $x = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-)(-)$</td>
<td>$(+)(-)$</td>
<td>$(+)(-)$</td>
<td>$(+)(+)$</td>
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<tr>
<td>$(-)$</td>
<td>$(-)$</td>
<td>$(+)$</td>
<td>$(+)$</td>
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<tr>
<td>$-$</td>
<td>$-$</td>
<td>$+$</td>
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The solution to the inequality $\frac{x^2 + 2x - 3}{x + 1} \geq 0$ is $-3 \leq x < -1$ or $x \geq 1$. 